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DYNAMIC SADDLE-NODE BIFURCATION IN A CLASS OF SLOW AND FAST PREDATOR-PREY MODELS

Hafida BOUDJELLABA[1] , Tewfik SARI[2]

ABSTRACT. We study the stability loss delay phenomenon in the dynamic saddle-node bifurcation in a class of three-dimensional prey and predator systems. The dynamics of the predator is assumed to be slow comparatively to the dynamics of the preys. As an application, a well-known model considered by Clark will be discussed.

AMS SUBJECT CLASSIFICATIONS: 34D15, 34E15, 92D25.
KEYWORDS Asymptotic stability, Delayed loss of stability, Singular perturbations, Canards, Biological models.

CONTENTS

1. INTRODUCTION

In most applications the dynamics of different variables of a system of ordinary differential equations are hierarchically scaled: for instance, in ecological models, often, the preys multiply much faster than the predators.

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Hence, the study and management of systems with various time scales were considered by many authors and remain a high point of interest both from theoretical and practical points of view [5, 6, 16, 19, 20]. Properties of solutions of such systems can be studied by using singular perturbation theory or Tikhonov's theory [21, 32, 34].

In singular perturbation theory, the *delayed loss of stability* phenomenon was first described through a typical example [28]. The general theory, for slow and fast systems under the assumption that the fast equation undergoes a Hopf bifurcation, have been extensively studied in the literature, we refer to [22, 23, 24] and for references and complements to [26, 29, 30]. An important aspect of the stability loss delay phenomenon is its close relation to the phenomenon of *canard* solutions (see [1], p. 179–192 and [2]). Canard solutions are special trajectories of slow and fast systems that first move near the stable part of the slow manifold, then move near the unstable part of it. They were first studied in the framework of Non-Standard Analysis [4, 11, 36] and then in the framework of geometric singular perturbation theory [13], using center manifolds and blow-up [10, 12, 17, 25, 31, 35].

The theory of the stability loss delay for planar systems is well known [10, 11, 12, 17]. In the case of three dimensional systems, the theory is much more difficult. When the system admits two slow variables and one fast variable, some results have been obtained, we refer to [3, 31, 35]. When the system has one slow variable and two fast variables, results have been published in the case of the dynamic Hopf bifurcation [22, 23, 24, 26] and in the case of bifurcations of periodic solutions [29, 30]. However, the situations where only one real eigenvalue of the fast system crosses zero, can be reduced locally to the planar case by center manifold reduction [17].

In this paper, we propose to study the delayed loss of stability in a class of slow and fast systems under the assumption that the fast equation undergoes a saddle node bifurcation. More precisely, we consider the following system of ordinary differential equations:

$$\begin{array}{rcl} x' & = & xM(x,y,E), \\ y' & = & yN(x,y,E), \\ E' & = & \varepsilon P(x,y,E), \end{array} \tag{1.1}$$

where M, N and P are smooth functions. We assume that ε is small , which means that reaction E is much slower than reactions x and y. We assume that the equilibrium $(0,0)$ of the fast equation corresponding to (1.1) is an unstable node for $0 \le E < a$, a saddle point for $a \le E < b$ and a stable node for $E \ge b$ where $b > a > 0$ are the bifurcation values at which the equilibrium $(0,0)$ collides with other equilibria of the system. System (1.1) admits $x = y = 0$ as a particular solution. This solution is a canard solution. Our main problem will be to describe the behaviour of the solutions of (1.1) in the vicinity of this canard solution : we calculate the *entrance-exit* function along the E-axis. A solution which jumps quickly near the e-axis with $E_0 > b$ will move near this axis, with decreasing E, until

E reaches a value $E_1 < b$. The mapping $E_0 \mapsto E_1$ is called the entrance-exit function along the canard solution $x = y = 0$. The center manifold reduction predicts that locally (see [17]), near the transcritical bifurcation for $E = b$, the entrance-exit function along the canard solution $x = y = 0$ of (1.1) is simply the entrance-exit function along the canard solution $y = 0$ of the reduction of system (1.1) to the invariant plane $x = 0$ (see Section 3.1). In this paper, we will calculate the global entrance-exit function $E_0 \mapsto E_1$ for all $E_0 > b$.

The usual saddle-node bifurcation is a collision and disappearance of a saddle and a node of a dynamical system [18]. In (1.1), the saddle-node bifurcation is degenerate since the equilibrium does not disappear and is persistent. We believe that this non generic saddle-node bifurcation deserves a special study since systems like system (1.1), known in mathematical biology as Kolmogorov systems, are often considered in the literature [14, 19, 27]. We impose biological meaningful conditions on the functions M, N, P such that the fast dynamics undergoes saddle-node bifurcations. We show that the system has a globally asymptotically stable (GAS) equilibrium and we describe the behavior of the trajectories towards this equilibrium. The results are illustrated in the Lotka-Volterra model of combined harvesting of two ecologically independent populations which was considered by Clark in his monography [9]:

$$\begin{array}{rcl} x' & = & rx(1 - x/K) - q_1 E x, \\ y' & = & sy(1 - y/L) - q_2 E y, \\ E' & = & \varepsilon\,(p_1 q_1 x + p_2 q_2 y - c)\,E. \end{array} \tag{1.2}$$

The parameters r and s are the intrinsic growth rates, q_1 and q_2 are the catchability coefficients and K and L are carrying capacities for populations x and y respectively, E is the harvesting effort, p_1 and p_2 are the prices and cE is the cost of fishing. Clark stated, without proof, that under some conditions system (1.2) has a persistent equilibrium $(x_\infty, y_\infty, E_\infty)$ which is approached asymptotically (see [9], p. 312). The dynamics of (1.2) have been investigated when ε is not a small parameter and the conditions for system (1.2) to have a GAS equilibrium are known [27, 33]. The main problem in the study of dynamical systems arising in applications is to determine the asymptotic behavior of the solutions. In mathematical ecology, this problem is related to the study of the persistence of the species [14]. For instance, if there exists a GAS steady state of positive coordinates, then the system is persistent.

The paper is organized as follows. In Section 2, we apply Tikhonov's theory to system (1.1). In Section 3, we study the delayed loss of stability phenomenon in this system. In Section 4, the transcient behavior of the trajectories towards the equilibrium is investigated. In Section 5, the results are illustrated on Clark's model (1.2) and by numerical simulations.

2. Fast Dynamics and Slow Dynamics

Let us denote by τ the time in system (1.1). In terms of the slow time $t = \varepsilon\tau$, system (1.1) becomes

$$\begin{array}{rcl} \varepsilon\dot{x} & = & xM(x,y,E) \\ \varepsilon\dot{y} & = & yN(x,y,E), \\ \dot{E} & = & P(x,y,E). \end{array} \tag{2.1}$$

Throughout the paper, the dot designates the derivatives with respect to time t and the prime designates the derivatives with respect to time τ. We assume that

(A0): $P(x,y,0) = 0$ for all $x \geq 0$ and $y \geq 0$ and $P(0,0,E) < 0$ for all $E > 0$.

We study (2.1) in the invariant non negative cone of $\mathbb{R}^3$

$$C^3 = \{(x,y,E) \in \mathbb{R}^3 : x \geq 0, y \geq 0, E \geq 0\}.$$

2.1. **The slow manifold.** System (2.1) is a slow and fast system, with E as the slow variable and x, y as the fast variables. For this system, the fast equations are written as

$$\begin{array}{rcl} x' & = & xM(x,y,E) \\ y' & = & yN(x,y,E) \end{array} \tag{2.2}$$

where E is a constant parameter. The following assumptions are made

(A1): The subset $\mathcal{S}_2 = \{(x,y,E) \in C : x = N(x,y,E) = 0\}$ is the graph of a smooth function $(x,y) = (0,\eta(E))$ where $0 \leq E \leq b$, $\eta(E) > 0$ for $0 \leq E < b$ and $\eta(b) = 0$.
The subset $\mathcal{S}_4 = \{(x,y,E) \in C : y = M(x,y,E) = 0\}$ is the graph of a smooth function $(x,y) = (\xi(E),0)$ where $0 \leq E \leq a < b$ such that $\xi(E) > 0$ for $0 \leq E < a$ and $\xi(a) = 0$.
The subset $\mathcal{S}_3 = \{(x,y,E) \in C : M(x,y,E) = N(x,y,E) = 0\}$ is the graph of a smooth function $(x,y) = (\xi_1(E),\eta_1(E))$ where $0 \leq E \leq c < b$ such that $\xi_1(E) > 0$, $\eta_1(E) > 0$ for $0 \leq E < c$ and $\xi_1(c) = 0$, $\eta_1(c) = \eta(c)$.

From assumption **(A1)** we see that the slow manifold

$$\mathcal{S} = \{(x,y,E) \in C : xM(x,y,E) = yN(x,y,E) = 0\},$$

which is the set of equilibra of (2.2) consists of four curves (see Fig. 1):

i) The curve $\mathcal{S}_1 = \{(0,0,E) \in C : E \geq 0\}$. On this slow curve, the slow equation is

$$\dot{E} = P(0,0,E), \qquad E \geq 0. \tag{2.3}$$

ii) The curve $\mathcal{S}_2 = \{(0,y,E) \in C : y = \eta(E), 0 \leq E \leq b\}$. On this slow curve, the slow equation is

$$\dot{E} = P(0,\eta(E),E), \qquad 0 < E < b. \tag{2.4}$$

iii) The curve $\mathcal{S}_3 = \{(x, y, E) \in C : x = \xi_1(E), y = \eta_1(E), 0 \le E \le c\}$. On this slow curve, the slow equation is

$$\dot{E} = P(\xi_1(E), \eta_1(E), E), \qquad 0 < E < c. \tag{2.5}$$

iv) The curve $\mathcal{S}_4 = \{(x, 0, E) \in C : x = \xi(E), 0 \le E \le a\}$. On this slow curve, the slow equation is

$$\dot{E} = P(\xi(E), 0, E), \qquad 0 < E < a. \tag{2.6}$$

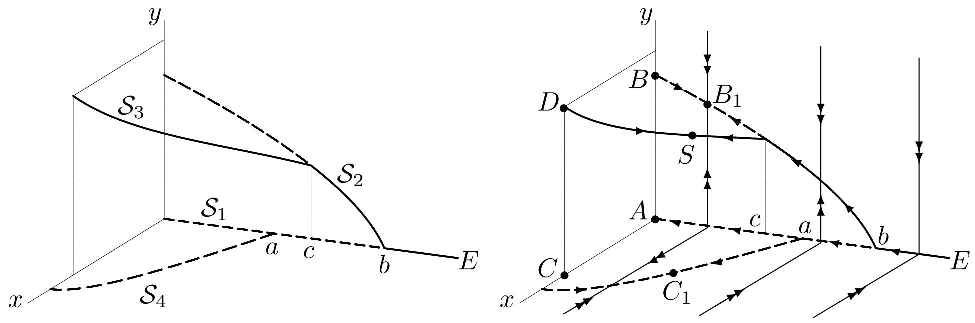

FIGURE 1. The slow manifold of system (2.1): attracting parts of the slow manifold are indicated by a bold line, non attracting parts of the slow manifold are indicated by a dashed line. On the left is the case $a \le c$. On the right is the case $a \ge c$ showing the fast dynamics (with two arrows) on the invariant planes $x = 0$ or $y = 0$, the slow dynamics (with one arrow) on the slow manifold and the equilibria A, B, B_1, C, C_1, D and S.

When $E \ge b$, $(0, 0)$ is the only equilibrium of (2.2); when $\max(a, c) \le E < b$, (2.2) has two equilibria, $(0, 0)$ and $(0, \eta(E))$; when $0 \le E < \min(a, c)$, (2.2) has four equilibria, $(0, 0)$, $(\xi(E), 0)$, $(0, \eta(E))$ and $(\xi_1(E), \eta_1(E))$. In the case where $c \le E < a$, (2.2) has three equilibria, $(0, 0)$, $(\xi(E), 0)$ and $(0, \eta(E))$. In the case where $a \le E < c$, (2.2) has three equilibria, $(0, 0)$, $(0, \eta(E))$ and $(\xi_1(E), \eta_1(E))$. The stability of these equilibria is summarized in the following assumption:

(A2): The equilibrium $(0, 0)$ is a stable node when $E \ge b$, a saddle point when $a \le E < b$ and an unstable node when $0 \le E < a$ respectively. The equilibrium $(0, \eta(E))$ is a stable node when $c \le E < b$ and a saddle point when $0 \le E < c$ respectively. The equilibrium $(\xi(E), 0)$ is a saddle point when $0 \le E < a$. The equilibrium $(\xi_1(E), \eta_1(E))$ is a stable node when $0 \le E < c$. The basin of attraction of all stable nodes is the positive cone $C^2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ of $\mathbb{R}^2$.

Hence, (2.2) has degenerate saddle-node bifurcations when E crosses values a, b and c. When E increases and crosses value a, the saddle $(\xi(E), 0)$

and the unstable node $(0,0)$ collide and become the saddle $(0,0)$. When E increases and crosses value c, the saddle $(0,\eta(E))$ and the stable node $(\xi_1(E),\eta_1(E))$ collide and become the stable node $(0,\eta(E))$. When E increases and crosses value b, the stable node $(0,\eta(E))$ and the saddle $(0,0)$ collide and become the stable node $(0,0)$. We add the following assumption:

(A3): There exists $E_\infty \in [0,c[$ such that $E = E_\infty$ is a GAS equilibrium of the slow equation (2.5). There exists $b_1 \in [0,c[$ such that $E = b_1$ is a GAS equilibrium of the slow equation (2.4). There exists $a_1 \in [0,a[$ such that $E = a_1$ is a GAS equilibrium of the slow equation (2.6).

Assumptions **(A0-A3)** are just biologically reasonable and are satisfied in Lotka-Voltera type systems. By **(A0)**, the population E of predators decreases if there is no prey ($x = 0 = y$). By **(A2)**, if the population of predator is large ($E > b$), then both preys are led to extinction; if the population of predator is moderate ($b > E > a$), then the prey x cannot survive but the prey y is persistent and converges towards the stable steady state $y = \eta(E)$; when the population of predators further decreases ($E < a$), both preys are persistent and converge towards the stable steady states $x = \xi_1(E)$, $y = \eta_1(E)$. To avoid complicated behaviours and to make possible a complete description of the transcient behaviour of trajectories towards the GAS equilibrium S of the system, we assumed, in **(A1)** and **(A3)**, that the set defined by $P = 0$ intersects the curves $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ at unique points B_1, S and C_1. These properties are satisfied by Clark's model.

In geometric singular perturbation theory (GSPT) [13, 15], system (2.1) is called the *slow system* and system (1.1) is called the *fast system*. In this paper the terminology of the classical singular perturbation theory [21, 32, 34] is adopted and we refer to (2.2), which is the limit of (1.1) when $\varepsilon \to 0$, as the fast equations and to (2.4,2.6,2.5,2.3), which are limits of (2.1) when $\varepsilon \to 0$, as the slow equations. Note that in GSPT, the slow manifold is not necessarily attracting, as in Tikhonov's theorem. In GSPT the results hold for the more general case of slow and fast systems for which the slow manifold is normally hyperbolic. However, in GSPT, the vector field must be smooth, not only continuous, as it is the case in Tikhonov's theory. GSPT shows that for small $\varepsilon > 0$, (2.1) has a locally invariant manifold which is $O(\varepsilon)$-close to the slow manifold as long as the slow manifold is normally hyperbolic. If this normal hyperbolicity of the slow manifold is violated (which is always the case at bifurcations of the slow manifolds), more delicate phenomena are expected.

2.2. **Equilibria of the system.** From assumption **(A3)**, point

$$S = (x_\infty, y_\infty, E_\infty) \in \mathcal{S}_3, \qquad \text{where } x_\infty = \xi_1(E_\infty), \quad y_\infty = \eta_1(E_\infty) \tag{2.7}$$

is an equilibrium of (2.1). In fact (2.1) has at least four other equilibria,

$$A = (0,0,0), \quad B = (0,\eta(0),0), \quad C = (\xi(0),0,0), \quad D = (\xi_1(0),\eta_1(0),0),$$

which lie in the invariant xy-plane. In the case where $b_1 > 0$

$$B_1 = (0, \eta(b_1), b_1) \in \mathcal{S}_2 \tag{2.8}$$

is an equilibrium of (2.1). It is a saddle point whose stable manifold is the invariant plane $x = 0$. When $b_1 = 0$ equilibria B and B_1 collide. In the case where $a_1 > 0$, (2.1) has another equilibrium

$$C_1 = (\xi(a_1), 0, a_1) \in \mathcal{S}_4. \tag{2.9}$$

It is a saddle point whose stable manifold is the invariant plane $y = 0$. When $a_1 = 0$ equilibria C and C_1 collide (see Fig. 1, right).

2.3. **Application of Tikhonov's theory.** From **(A2)**, for all $E \in [0, c[$, the equilibrium $(\xi_1(E), \eta_1(E))$ of (2.2) is GAS in C^2. Moreover, by **(A3)**, system (2.5) has a GAS equilibrium point $E = E_\infty$. Hence, Tikhonov's theory applies and predicts that in the region $0 \le E < c$, the solutions of (2.1) jump quickly near the slow curve $\mathcal{S}_3$ and then move near this slow curve towards S (see Proposition 4.1).

From **(A2)**, for all $E \in [c, b[$, the equilibrium $(0, \eta(E))$ of (2.2) is GAS in C^2. Hence, Tikhonov's theory applies and predicts that in the region $c \le E < b$, the solutions of (2.1) jump quickly near the slow curve $\mathcal{S}_2$ and then move near this slow curve with decreasing E, until E reaches the value c at which this slow curve loses its stability. One might believe then, that the solution will move, for $E < c$, near the attracting slow curve $\mathcal{S}_3$, towards S (see Fig. 2, left). In fact, due to the delayed loss of stability phenomenon, this behavior is not the right one and the solution will stay near the slow curve $\mathcal{S}_2$, until E reaches a value $E_1 < c$, (see Fig. 5, right).

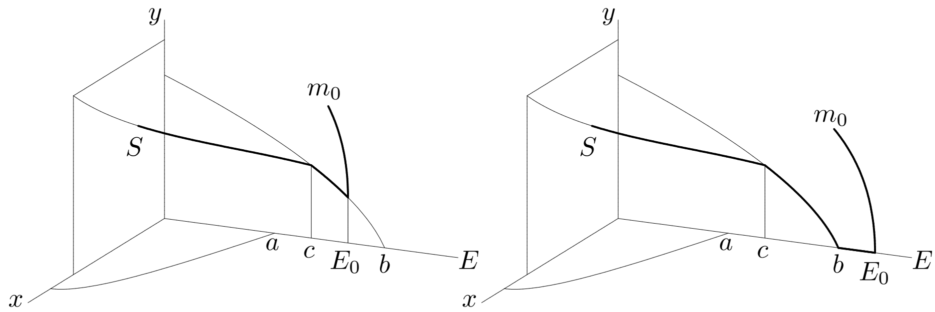

FIGURE 2. The asymptotic behavior of the solution of (2.1) with initial condition $m_0 = (x_0, y_0, E_0)$ when $c < E_0 < b$ (on the left) and $E_0 > b$ (on the right). The expected asymptotic behavior, when E crosses the value c (on the left) or the values b and c (on the right) is not correct. The correct behavior, showing the delayed loss of stability, is described in Fig. 4, 5, 6 or 7.

From **(A2)**, for all $E \ge b$, the equilibrium $(0, 0)$ of (2.2) is GAS in C^2. Again, Tikhonov's theory applies and predicts that in the region $E > b$, the solutions of (2.1) jump quickly near the slow curve $\mathcal{S}_1$ and then move near this axis with decreasing E, until E reaches the value b at which this slow curve loses its stability (see Fig. 2, right). One might believe then, that the solution will move, for $E < b$, near the attracting slow curve $\mathcal{S}_2$, with decreasing E, until E reaches the value c and then will move, for $E < c$, near the attracting slow curve $\mathcal{S}_3$, towards S (see Fig. 2, right). In fact, due to the delayed loss of stability phenomenon, this behavior is not the right one and the solution will stay near the slow curve $\mathcal{S}_1$ until E reaches a value $E_1 < b$ (see Fig. 4, 6 or 7, right).

3. Delayed loss of stability

The mapping $E_0 \mapsto E_1$ (see Fig. 5) is called the *entrance-exit* function along the slow curve $\mathcal{S}_2$. Similarly, the mapping $E_0 \mapsto E_1$ (see Fig. 4, 6 or 7) is called the *entrance-exit function* along the slow curve $\mathcal{S}_1$. Our aim in this section is to calculate these entrance-exit function. As mentioned in the introduction, locally near the transcritical bifurcations for $E = a, b, c$, the entrance-exit functions are obtained by a center manifold reduction. Throughout the paper we denote by

$$\mu(E) = \frac{M(0,0,E)}{P(0,0,E)}, \qquad \nu(E) = \frac{N(0,0,E)}{P(0,0,E)},$$
$$\lambda(E) = \frac{M(0,\eta(E),E)}{P(0,\eta(E),E)}, \qquad \kappa(E) = \frac{N(\xi(E),0,E)}{P(\xi(E),0,E)}.$$

The functions μ and ν are defined for all $E > 0$. The function λ is defined for $E \in]b_1, b[$. The function κ is defined for $E \in]a_1, a[$.

3.1. Center manifold reduction.

The invariant plane $y = 0$ is a center manifold of (2.1) at $(x, y, E, \varepsilon) = (0, 0, a, 0)$. On this center manifold (2.1) reduces to the slow-fast planar system

$$\varepsilon\dot{x} = xM(x,0,E), \qquad \dot{E} = P(x,0,E). \tag{3.1}$$

The particular solution $x = 0$ of this system is a canard solution. Let (x_0, E_0) be an initial condition such that $E_0 > a$. From the theory of canard solutions in planar slow-fast vector fields, the corresponding solution of (3.1) will go quickly towards $x = 0$ and then remains close to this canard solution while E is decreasing and until E reaches a value $E_1 = H(E_0) < a$. Then it jumps to the neighborhood of the slow curve $\mathcal{S}_4$ and converges towards the equilibrium (2.9). The entrance-exit function $E_0 \mapsto E_1 = H(E_0)$ along the canard solution $x = 0$ of (3.1) is calculated as follows (see [11], Formula VI). Let $h(E) = \int_a^E \mu(u)du$. This function has a minimum at a (see Fig. 3, right). It is decreasing from $+\infty$ to 0 on $]0, a]$ and increasing on $[a, +\infty[$. It defines a mapping $H = h_+^{-1} \circ h_- : [a, \infty[\to]0, a]$, where h_- and h_+ are the restrictions of h on $[a, +\infty[$ and $]0, a]$ respectively. We have

$$\int_E^{H(E)} \mu(u)du = 0. \tag{3.2}$$

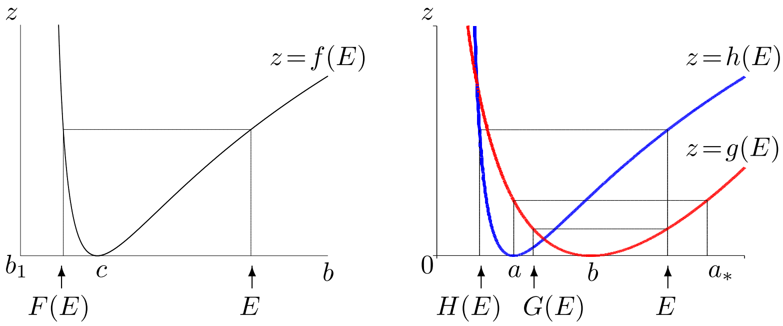

FIGURE 3. On the left: the function f defining the entrance-exit function $E \mapsto F(E)$ along the slow curve $\mathcal{S}_2$. On the right: the functions g and h defining the mappings $E \mapsto G(E)$ and $E \mapsto H(E)$ respectively.

Similarly, the invariant plane $x = 0$ is a center manifold of (2.1) at $(x, y, E, \varepsilon) = (0, 0, b, 0)$. On this center manifold (2.1) reduces to the slow-fast planar system

$$\varepsilon \dot{y} = yN(0, y, E), \qquad \dot{E} = P(0, y, E). \tag{3.3}$$

The particular solution $y = 0$ of this system is a canard solution. Let (y_0, E_0) be an initial condition such that $E_0 > b$. The corresponding solution of (3.3) will go quickly towards $y = 0$ and then remains close to this canard solution while E is decreasing and until E reaches a value $E_1 = G(E_0) < b$. Then it jumps to the neighborhood of the slow curve $\mathcal{S}_2$ and converges towards the equilibrium (2.8). The entrance-exit function $E_0 \mapsto E_1 = G(E_0)$ along the canard solution $y = 0$ of (3.3), is calculated as follows. The function $g = \int_b^E \nu(u)du$ has a minimum at b (see Fig. 3, right). It is decreasing from $+\infty$ to 0 on $]0, b]$ and increasing on $[b, +\infty[$. It defines a mapping $G = g_+^{-1} \circ g_- : [b, \infty[\to]0, b]$ where g_- and g_+ are the restrictions of g on $[b, +\infty[$ and $]0, b]$ respectively. We have

$$\int_E^{G(E)} \nu(u)du = 0. \tag{3.4}$$

Let $a_* = G^{-1}(a)$. If $E \in [b, a_*]$ then (see Fig. 3, right) $H(E) < a \leq G(E)$. If $E > a_*$ then, we either have $H(E) < G(E)$ or $H(E) > G(E)$, see Section 3.5. However, in (1.2), the property $H(E) < G(E)$ is true for all $E > a_*$, see Lemma 5.1.

For all $y \in [0, b[$, the curve $y = \eta(E)$ is an attracting hyperbolic slow curve of (3.3). Hence this system admits an invariant curve $y = \eta(E, \varepsilon)$ such that $\eta(E, 0) = \eta(E)$, defined for $E \in [0, b_2] \subset [0, b[$. Notice that b_2 can be chosen as close to b as we want. This invariant curve belongs to a center manifold $y = \eta(x, E, \varepsilon)$, such that $\eta(0, E, \varepsilon) = \eta(E, \varepsilon)$. On this center manifold (2.1) reduces to

$$\varepsilon \dot{x} = xM(x, \eta(x, E, \varepsilon), E), \qquad \dot{E} = P(x, \eta(x, E, \varepsilon), E). \tag{3.5}$$

The particular solution $x = 0$ of this system is a canard solution. Let (x_0, E_0) be an initial condition such that $b > E_0 > c$. The corresponding solution of (3.5) will go quickly towards $x = 0$ and then remains close to this canard solution while E is decreasing and until E reaches a value $E_1 = F(E_0) < c$. We calculate the entrance-exit function $E_1 \mapsto E_0 = G(E_1)$ along the canard solution $x = 0$ of (3.5) as follows. The function $f(E) = \int_c^E \lambda(u)du$, is defined for all $E \in]b_1, b]$. The function f reaches its minimum at c (see Fig. 3, left). It is decreasing from $+\infty$ to 0 on $]b_1, c]$ and increasing on $[c, b]$. It defines a mapping $F = f_+^{-1} \circ f_- : [c, b[\to]b_1, c]$, where f_- and f_+ are the restrictions of f on $[c, b[$ and $]b_1, c]$. We have

$$\int_E^{F(E)} \lambda(u)du = 0. \tag{3.6}$$

From the previous analysis we deduce the following result:

Proposition 3.1. *The trajectories*

$$\mathcal{C}_1 = \{(0, 0, E) : E > 0\} \quad \textit{and} \quad \mathcal{C}_2 = \{(0, \eta(E, \varepsilon), E) : b_1 < E < b_2\}$$

are canard solutions of (2.1).

Notice that $\mathcal{C}_1 \subset \mathcal{S}_1$ and $\mathcal{C}_2$ is near $\mathcal{S}_2$. Let $\gamma(t, \varepsilon) = (x(t, \varepsilon), y(t, \varepsilon), E(t, \varepsilon))$ the solution of (2.1) with initial condition $x(0, \varepsilon) = x_0 > 0$, $y(0, \varepsilon) = y_0 > 0$, and $E(0, \varepsilon) = E_0 > 0$. We assume that the solution approaches one of the canard solutions $\mathcal{C}_1$ or $\mathcal{C}_2$. The main problem is to calculate the entrance-exit function of (2.1) along the canard solution. In the context of GSPT, the problem is completely solved locally (see [17]) using a center manifold reduction. Hence, locally near the transcritical bifurcation at $E = a, b, c$, the entrance-exit functions of the three dimensional system (2.1) are simply the entrance-exit functions (3.2), (3.4) and (3.6) of the two dimensional systems (3.1), (3.3) and (3.5) respectively. More precisely, we have the following result:

Theorem 3.2. *There exists $\delta > 0$, such that for small $\varepsilon > 0$, we have*
i) if $E_0 \in]c, \min(b, c + \delta)[$ then the solution $\gamma(t, \varepsilon)$ remains near the slow curve $\mathcal{S}_2$ as long as $E_1 < E < E_0$ where $E_1 = F(E_0)$ and jumps to the neighborhood of point $(\xi_1(E_1), \eta_1(E_1), E_1)$ close to the unstable separatrix of the saddle point $(0, \eta(E_1))$ of the fast dynamics.
ii) if $E_0 \in]b, b + \delta[$ then the solution $\gamma(t, \varepsilon)$ remains near the slow curve $\mathcal{S}_1$ as long as $E_1 < E < E_0$ where $E_1 = G(E_0)$ and jumps to the neighborhood of point $(0, \eta(E_1), E_1)$ close to the unstable separatrix $x = 0$ of the saddle point $(0, 0)$ of the fast dynamics.
iii) if $E_0 \in]\max(0, a - \delta), a[$ then the solution $\gamma(t, \varepsilon)$ remains near the slow curve $\mathcal{S}_1$ as long as $E_0 < E < E_{-1}$ where $E_{-1} = H^{-1}(E_0)$ and jumps (for reversed time) far from $(0, 0, E_{-1})$ along the stable separatrix $y = 0$ of the saddle point $(0, 0)$ of the fast dynamics.

In the following section we show how to solve this problem globally.

3.2. **Behavior in the vicinity of the slow curve $\mathcal{S}_1$.** Let (x_0, y_0, E_0) be an initial condition such that $0 < x_0 < 1$, $0 < y_0 < 1$ and $E_0 > b$. If ε is small enough, the corresponding trajectory $\gamma(t, \varepsilon)$ of (2.1) remains in the domain $0 < x < x_0$ and $0 < y < y_0$ and goes towards $\mathcal{S}_2$ while E is decreasing as far as $E > b$. Denote now, the next intersection of this trajectory and the planes $x = x_0$ or $y = y_0$ by $(x(t_1, \varepsilon), y(t_1, \varepsilon), E(t_1, \varepsilon))$ where $t_1 = t_1(x_0, y_0, E_0, \varepsilon)$ depends on the initial condition (x_0, y_0, E_0) and on ε. We have

$$\lim_{\varepsilon \to 0} E(t_1(x_0, y_0, E_0, \varepsilon), \varepsilon) = K(E_0) := \max(G(E_0), H(E_0))$$

We have already noticed that if $b < E < a^*$ then $G(E) > H(E)$, so this result is in agreement with the local one given by Theorem 3.2, case ii. More precisely, we have the following result:

Proposition 3.3. *For small $\varepsilon > 0$, the solution remains near the slow curve $\mathcal{S}_1$ as long as $E_1 < E < E_0$ where $E_1 = K(E_0)$.*
i) If $E_0 \in [b, a_*]$ or $E_0 > a_*$ and $H(E_0) < G(E_0)$, then the solution leaves the neighborhood of point $(0, 0, E_1)$ and jumps to the neighborhood of point $(0, \eta(E_1), E_1)$ close to the straight line $x = 0$ (see Fig. 4 and 6).
ii) If $E_0 > a_*$ and $H(E_0) > G(E_0)$ then, the solution leaves the neighborhood of point $(0, 0, E_1)$ and jumps to the neighborhood of point $(\xi(E_1), 0, E_1)$ close to the straight line $y = 0$ (see Fig. 7).

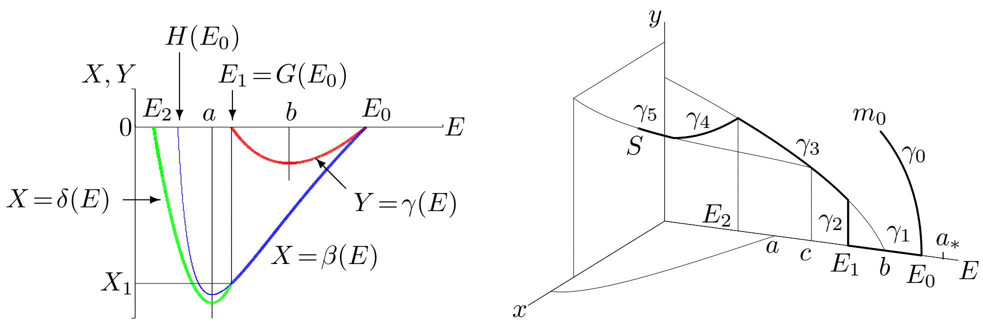

FIGURE 4. On the right: the asymptotic behavior of the solution of (2.1) with initial condition $m_0 = (x_0, y_0, E_0)$, when $b < E_0 < a_*$, showing the delayed loss of stability when E crosses values b and c. On the left: the asymptotic behavior in the coordinates (X, Y, E) of (3.7). The functions β (in blue), γ (in red) and δ (in green) are defined by (3.9) and (3.17) respectively.

Proof. Let $x_0 < 1$ and $y_0 < 1$ be positive and not too big. For small $\varepsilon > 0$ the solution remains in the domain $0 < x < x_0$ and $0 < y < y_0$. The change of variables $X = \varepsilon \ln x$, $Y = \varepsilon \ln y$ maps the open set $0 < x < 1$, $0 < y < 1$

into the octant $X < 0$, $Y < 0$. This change of variables transforms (2.1) into

$$\begin{array}{rcl} \dot{X} & = & M(\exp(X/\varepsilon), \exp(Y/\varepsilon), E), \\ \dot{Y} & = & N(\exp(X/\varepsilon), \exp(Y/\varepsilon), E), \\ \dot{E} & = & P(\exp(X/\varepsilon), \exp(Y/\varepsilon), E). \end{array} \tag{3.7}$$

The initial condition (x_0, y_0, E_0) becomes $(\varepsilon \ln x_0, \varepsilon \ln y_0, E_0)$. Since $X < 0$ and $Y < 0$, (3.7) is a regular perturbation of

$$\dot{X} = M(0,0,E), \qquad \dot{Y} = N(0,0,E), \qquad \dot{E} = P(0,0,E). \tag{3.8}$$

The solution of (3.8) with initial condition $X(0) = 0$, $Y(0) = 0$, $E(0) = E_0$ is given by $E = \overline{E}(t)$ and

$$X = \beta(E) := \int_{E_0}^{E} \mu(u)du, \qquad Y = \gamma(E) := \int_{E_0}^{E} \nu(u)du, \tag{3.9}$$

where $\overline{E}(t)$ is the solution of (2.3) such that $\overline{E}(0) = E_0$. Let $E_1 = K(E_0)$ and t_1 satisfies $\overline{E}(t_1) = E_1$. Hence $E(t_1, \varepsilon) = E_1 + o(1)$. When E is asymptotically equal to E_1, i.e. when t is asymptotically equal to t_1, the solution jumps far from the neighborhood of the E-axis as shown below.

Case i. If $E_0 \in [b, a_*]$ (see Fig. 4, left), or $E_0 > a_*$ and $H(E_0) < G(E_0)$ (see Fig. 6, left), then $E_1 = G(E_0)$. Thus, according to (3.4) and $h(G(E_0)) < h(E_0)$, we have

$$X_1 = \int_{E_0}^{E_1} \mu(u)du < 0, \qquad Y_1 = \int_{E_0}^{E_1} \nu(u)du = 0. \tag{3.10}$$

Since $X(t_1, \varepsilon) = X_1 + o(1)$ and $Y(t_1, \varepsilon) = o(1)$, the solution reaches again the plane $y = y_0$ asymptotically at time t_1, and $x(t_1, \varepsilon) = \exp((X_1 + o(1))/\varepsilon)$ is exponentially small. Thus, asymptotically at time t_1, the solution jumps (see Fig. 4 or 6, right) from the neighborhood of point $(0, 0, E_1)$ to the neighborhood of point $(0, \eta(E_1), E_1)$ close to the straight line $x = 0$.

Case ii. If $E_0 > a_*$ and $H(E_0) > G(E_0)$ (see Fig. 7, left) then $E_1 = H(E_0)$. Thus, according to (3.2) and $g(H(E_0)) < g(E_0)$, we have

$$X_1 = \int_{E_0}^{E_1} \mu(u)du = 0, \qquad Y_1 = \int_{E_0}^{E_1} \nu(u)du < 0. \tag{3.11}$$

Since $X(t_1, \varepsilon) = o(1)$ and $Y(t_1, \varepsilon) = Y_1 + o(1)$, the solution reaches again the plane $x = x_0$ asymptotically at value E_1 and $y(t_1, \varepsilon) = \exp((Y_1 + o(1))/\varepsilon)$ is exponentially small. Thus, asymptotically at time t_1, the solution jumps (see Fig. 7, right) from the neighborhood of point $(0, 0, E_1)$ to the neighborhood of point $(\xi(E_1), 0, E_1)$ close to the straight line $y = 0$. $\square$

3.3. **Behavior in the vicinity of the slow curve $\mathcal{S}_2$.** Let (x_0, y_0, E_0) be an initial condition such that $0 < x_0 < 1$ and $c < E_0 < b$. If ε is small enough, the corresponding trajectory $\gamma(t, \varepsilon)$ of (2.1) remains between the planes $x = x_0$ and $x = 0$ and goes towards $\mathcal{S}_2$ while E is decreasing as far as $E > c$. Denote now, the next intersection of this trajectory and the plane

$x = x_0$ by $(x_0, y(t_1, \varepsilon), E(t_1, \varepsilon))$ where $t_1 = t_1(x_0, y_0, E_0, \varepsilon)$ depends on the initial condition (x_0, y_0, E_0) and on ε. We have

$$\lim_{\varepsilon \to 0} E(t_1(x_0, y_0, E_0, \varepsilon), \varepsilon) = F(E_0).$$

Thus, the result of Theorem 3.2, case i, holds for all $c < E_0 < b$, not only for $c < E_0 < \min(b, c + \delta)$. More precisely, we have the following result:

Theorem 3.4. *Let $E_0 \in]a, b[$. The solution remains near the slow curve $\mathcal{S}_2$ as long as $E_1 < E < E_0$ where $E_1 = F(E_0)$ and jumps to the neighborhood of point $(\xi_1(E_1), \eta_1(E_1), E_1)$ close to the unstable separatrix of the saddle point $(0, \eta(E_1))$ of the fast dynamics (see Fig. 5).*

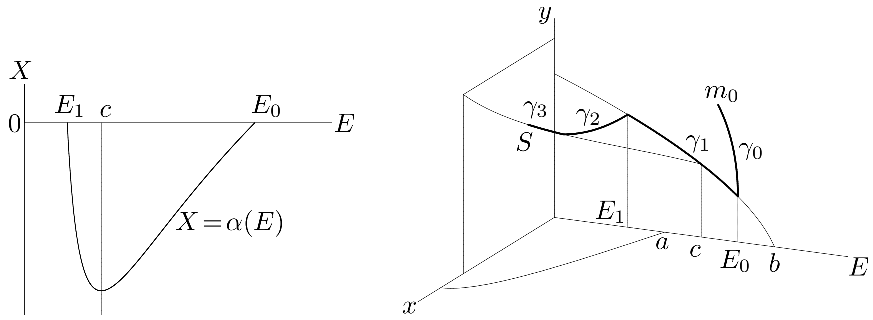

Figure 5. On the right: the asymptotic behavior of the solution of (2.1) with initial condition $m_0 = (x_0, y_0, E_0)$, when $c < E_0 < b$, showing the delayed loss of stability when E crosses the value c. On the left: the asymptotic behavior in the coordinates (X, E) of (3.12). The function α is defined by (3.15).

Proof. The change of variable $X = \varepsilon \ln x$ maps the strip $0 < x < 1$ into the half space $X < 0$. This change of variable transforms (2.1) into

$$\begin{array}{rcl} \dot{X} & = & M(\exp(X/\varepsilon), y, E), \\ \varepsilon \dot{y} & = & yN(\exp(X/\varepsilon), y, E) \\ \dot{E} & = & P(\exp(X/\varepsilon), y, E) \end{array} \tag{3.12}$$

The initial condition becomes $(\varepsilon \ln x_0, y_0, E_0)$. System (3.12) is a slow and fast system, with X, E as the slow variables and y as the fast variable. We have $\lim_{\varepsilon \to 0} \exp X/\varepsilon = 0$ since $X < 0$. Thus, the fast equation is written as

$$y' = yN(0, y, E) \tag{3.13}$$

The equilibrium $y = \eta(E)$ of (3.13) is attracting for all $E \in [0, b[$. Thus, on the slow surface $y = \eta(E)$, the slow equation is

$$\dot{X} = M(0, \eta(E), E), \qquad \dot{E} = P(0, \eta(E), E). \tag{3.14}$$

According to Tikhonov's theory, y goes very quickly towards the stable equilibrium $y = \eta(E)$. Then a slow transition develops near the surface $y = \eta(E)$. This slow transition is approximated by the solution of (3.14) with initial condition $X(0) = 0$, $E(0) = E_0$. This solution is given by $E = \overline{E}(t)$ and

$$X = \alpha(E) := \int_{E_0}^{E} \lambda(u)du, \tag{3.15}$$

where $\overline{E}(t)$ is the solution of (2.4) such that $\overline{E}(0) = E_0$. Thus, according to (3.6), we have again (see Fig. 5, left) $X = 0$ for $E_1 = F(E_0)$. Returning to the original variables, we see that the trajectory $\gamma(t, \varepsilon)$ crosses again the plane $x = x_0$ when E is asymptotically equal to $E_1 = F(E_0)$. Then (see Fig. 5, right), a fast transition brings the trajectory from the neighborhood of point $(0, \eta(E_1), E_1)$ to the neighborhood of point $(\xi(E_1), \eta(E_1), E_1)$ close to the unstable separatrix of the saddle point $(0, \eta(E_1))$ of the fast dynamics. □

As stated in Proposition 3.3, the jump of the solution, far from the slow curve $\mathcal{S}_1$, happens when E is asymptotically equal to $E_1 = K(E_1)$. After this jump, the asymptotic behavior of the solution is given by Theorems 3.5 and 3.6 below, where $b_0 = G^{-1}(b_1)$ if $b_1 > 0$ and $b_0 = \infty$ if $b_1 = 0$, $a_0 = H^{-1}(a_1)$ if $a_1 > 0$ and $a_0 = \infty$ if $a_1 = 0$.

Theorem 3.5. *Let $E_0 \in [b, a_*]$ or $E_0 > a_*$ and $H(E_0) < G(E_0)$. For small $\varepsilon > 0$, the solution remains near the slow curve $\mathcal{S}_1$ as long as $E_1 < E < E_0$ and near the slow curve $\mathcal{S}_2$ as long as $E_2 < E < E_1$ where $E_1 = G(E_0)$ and E_2 is defined as follows: if $E_0 = b_0$ then $E_2 = E_1 = b_1$; if $E_0 < b_0$ (resp. $E_0 > b_0$) then $E_2 \in]b_1, a[$ (resp. $E_2 \in]0, b_1[$) and E_2 is given by*

$$\int_{E_0}^{E_1} \mu(E)dE + \int_{E_1}^{E_2} \lambda(E)dE = 0. \tag{3.16}$$

Afterwards, it jumps from the neighborhood of point $(0, \eta(E_2), E_2)$ to the neighborhood of point $(\xi_1(E_2), \eta_1(E_2), E_2)$ close to the unstable separatrix of the saddle point $(0, \eta(E_2))$ of the fast dynamics (see Fig. 4 and 6).

Proof. The asymptotic behavior of the solution for $t \in [0, t_1]$ is described in Proposition 3.3, case i. For $t > t_1$ we use, the same change of variable as in the proof of Theorem 3.4, $X = \varepsilon \ln x$ which maps the strip $0 < x < 1$ into the half space $X < 0$. This change of variable transforms (2.1) into (3.12) with conditions $X(t_1, \varepsilon) = X_1 + o(1)$, $Y(t_1, \varepsilon) > 0$, and $E(t_1, \varepsilon) = E_1 + o(1)$, where X_1 is defined by (3.10). According to Tikhonov's theory, y goes very quickly towards the stable equilibrium $y = \eta(E)$. Then, a slow transition develops near the surface $y = \eta(E)$. This slow transition is approximated by the solution of (3.14) with initial condition $X(t_1) = X_1$, $E(t_1) = E_1$. If $E_1 \neq b_1$, this solution is given by

$$X = \delta(E) := X_1 + \int_{E_1}^{E} \lambda(u)du, \tag{3.17}$$

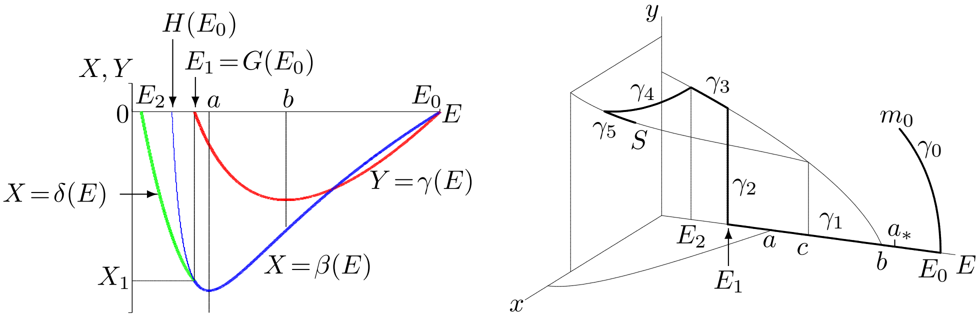

FIGURE 6. On the right: the asymptotic behavior of the solution of (2.1) with initial condition $m_0 = (x_0, y_0, E_0)$, when $E_0 > a_*$ and $H(E_0) < G(E_0)$, showing the delayed loss of stability when E crosses values b and c. On the left: the asymptotic behavior in the coordinates (X, Y, E) of (3.7). The functions β (in blue), γ (in red) and δ (in green) are defined by (3.9) and (3.17) respectively.

where $E = \overline{E}(t)$ is the solution of (2.4) such that $\overline{E}(t_1) = E_1$. In the case where $E_0 \in [b_0, a_*]$, E_2 defined by (3.16) satisfies $E_2 \in]b_1, a[$. And when $E_0 \in [b, b_0]$, E_2 defined by (3.16) satisfies $E_2 \in]0, b_1[$ (see Fig. 4, left). We have again $X = 0$ at value E_2. Returning to the original variables, we notice that the trajectory $\gamma(t, \varepsilon)$ crosses again the plane $x = x_0$ when E is asymptotically equal to E_2. Then (see Fig. 4, right), a fast transition brings the trajectory from the neighborhood of point $(0, \eta(E_2), E_2)$ to the neighborhood of point $(\xi_1(E_2), \eta_1(E_2), E_2)$, close to the unstable separatrix of the saddle point $(0, \eta(E_2))$ of the fast dynamics.
When $E_0 > a_*$ and $H(E_0) < G(E_0)$ the proof is similar to the case where $E_0 \in [b, a_*]$, adapted to Fig. 6. Now $E_1 \in]0, a[$ and E_2, given by (3.16), satisfies $E_2 \in]0, E_1[$ (see Fig. 6, left). □

Note that the mapping $E_1 \to E_2$ given by formula (3.16) is not equal to the entrance-exit function $E_1 \mapsto F(E_1)$ of the slow curve $\mathcal{S}_2$ as it was the case in Theorem 3.4. Indeed, in Theorem 3.5, the solution is exponentially close to the plane $x = 0$ before it arrives near the slow curve $\mathcal{S}_2$. Recall that in Theorem 3.4 the solution arrived from a point (x_0, y_0, E_0) which was not very close to the plane $x = 0$.

3.4. **Behavior in the vicinity of the slow curve $\mathcal{S}_4$.**

Theorem 3.6. *Let $E_0 > a_*$. If $H(E_0) > G(E_0)$ then, for small $\varepsilon > 0$, the solution remains near the slow curve $\mathcal{S}_1$ as long as $E_1 < E < E_0$ and near the slow curve $\mathcal{S}_4$ as long as $E_2 < E < E_1$ where $E_1 = H(E_0)$ and E_2 is defined as follows: if $E_0 = a_0$ then $E_2 = E_1 = a_1$; if $E_0 < a_0$ (resp.*

$E_0 > a_0$) then $E_2 \in]a_1, E_1[$ (resp. $E_2 \in]0, a_1[$) and E_2 is given by

$$\int_{E_0}^{E_1} \nu(E)dE + \int_{E_1}^{E_2} \kappa(E)dE = 0. \tag{3.18}$$

Afterwards, it jumps from the neighborhood of point $(\xi(E_2), 0, E_2)$ to the neighborhood of point $(\xi_1(E_2), \eta_1(E_2), E_2)$ close to the unstable separatrix of the saddle point $(\xi(E_2), 0)$ of the fast dynamics (see Fig. 7).

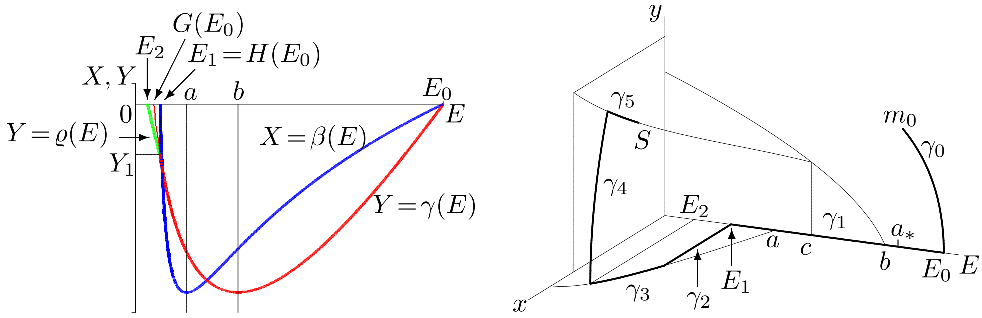

Figure 7. On the right: the asymptotic behavior of the solution of (2.1) with initial condition $m_0 = (x_0, y_0, E_0)$, in the case where $E_0 > a_*$ and $H(E_0) > G(E_0)$, showing the delayed loss of stability when E crosses values b and c. On the left: the asymptotic behavior in the coordinates (X, Y, E) of (3.7). The functions β (in blue), γ (in red) and ρ (in green) are defined by (3.9) and (3.22) respectively.

Proof. The asymptotic behavior of the solution for $t \in [0, t_1]$ is described in Proposition 3.3, case ii. For $t > t_1$ we use the change of variable $Y = \varepsilon \ln x$ which maps the strip $0 < y < 1$ into the half space $Y < 0$. This change of variable transforms (2.1) into

$$\begin{array}{rcl} \varepsilon\dot{x} & = & xM(x, \exp(Y/\varepsilon), E), \\ \dot{Y} & = & N(x, \exp(Y/\varepsilon), E), \\ \dot{E} & = & P(x, \exp(Y/\varepsilon), E) \end{array} \tag{3.19}$$

with conditions $X(t_1, \varepsilon) > 0$, $Y(t_1, \varepsilon) = Y_1 + o(1)$, and $E(t_1, \varepsilon) = E_1 + o(1)$, where Y_1 is defined by (3.11). System (3.19) is a slow and fast system, with Y, E as the slow variables and x as the fast variable. We have $\lim_{\varepsilon\to 0} \exp Y/\varepsilon = 0$ since $Y < 0$. Thus, the fast equation is written as

$$x' = xM(x, 0, E) \tag{3.20}$$

The equilibrium $x = \xi(E)$ of (3.13) is attracting for all $E \in [0, a[$. And on the slow surface $x = \xi(E)$, the slow equation is

$$\dot{Y} = N(\xi(E), 0, E), \qquad \dot{E} = P(\xi(E), 0, E). \tag{3.21}$$

According to Tikhonov's theory, x goes very quickly towards the stable equilibrium $x = \xi(E)$. Then a slow transition develops near the surface $x = \xi(E)$. This slow transition is approximated by the solution of (3.21) with initial condition $Y(t_1) = Y_1$, $E(t_1) = E_1$. If $E_1 \neq a_1$, this solution is given by

$$Y = \varrho(E) = Y_1 + \int_{E_1}^{E} \kappa(u)du, \tag{3.22}$$

where $E = \overline{E}(t)$ is the solution of (2.6) such that $\overline{E}(t_1) = E_1$. If $E_0 < a_0$ (resp. $E_0 > a_0$), then E_2 given by (3.18) satisfies $E_2 \in]a_1, E_1[$ (resp. $E_2 \in]0, a_1[$) (see Fig. 4, left). We have again $Y = 0$ at value E_2. Returning to the original variables, we see that the trajectory $\gamma(t, \varepsilon)$ crosses again the plane $y = y_0$ when E is asymptotically equal to E_2. Then (see Fig. 7, right), a fast transition brings the trajectory from the neighborhood of point $(\xi(E_2), 0, E_2)$ to the neighborhood of point $(\xi(E_2), \eta(E_2), E_2)$, close to the unstable separatrix $x = \xi(E_2)$ of the saddle point $(\xi(E_2), 0)$ of the fast dynamics. $\square$

3.5. **Example.** Notice that Assumptions **(A0-A3)** are not necessary to prove Proposition 3.3. The results stated in this proposition require only the following hypothesis : for all E, $P(0,0,E) < 0$ and there exist $b > a > 0$ such that

$$\forall E \neq a, \quad (E-a)M(0,0,E) < 0 \quad \text{and} \quad \forall E \neq b, \quad (E-b)N(0,0,E) < 0.$$

We can further divide system (1.1) by $P(x,y,E)$ and consider the system

$$x' = x\, m(x,y,t,\varepsilon), \qquad y' = y\, n(x,y,t,\varepsilon), \qquad t' = \varepsilon. \tag{3.23}$$

We denote $\mu(t) = m(0,0,t,0)$ and $\nu(t) = n(0,0,t,0)$ and we assume that there exist $a < b$ such that

$$\forall t \neq a, \quad (t-a)\mu(t) > 0 \quad \text{and} \quad \forall t \neq b, \quad (t-b)\nu(t) > 0.$$

Let H and G be the involutions defined by

$$\int_t^{H(t)} \mu(u)du = 0, \quad \text{and} \quad \int_t^{G(t)} \nu(u)du = 0.$$

The mapping $t \mapsto H(t)$ is the entrance-exit function along the canard solution $x = 0$ of (3.23) reduced to the invariant plane $y = 0$. The mapping $t \mapsto G(t)$ is the entrance-exit function along the canard solution $y = 0$ of (3.23) reduced to the invariant plane $x = 0$. We have the following theorem whose proof is very similar to the proof of Proposition 3.3 and is left to the reader.

Theorem 3.7. *For small $\varepsilon > 0$, the solution of (3.23) with initial condition $x(t_0) = x_0 > 0$ and $y(t_0) = y_0 > 0$ remains near the canard solution $x = y = 0$ as long as $t_0 < t < t_1$ where $t_1 = \min(H(t_0), G(t_0))$.*
a) If $H(t_0) < G(t_0)$ then the solution leaves the neighborhood of point $(0,0,t_1)$ close to the orbit $y = 0$ of the fast dynamics.

b) If $H(t_0) > G(t_0)$ then the solution leaves the neighborhood of point $(0,0,t_1)$ close to the orbit $x=0$ of the fast dynamics.

In the case where $H(t_0)=G(t_0)$, our analysis does not predict the orbit along which the fast transition far from $(0,0,t_1)$ should hold. By continuous dependence, all orbits of the fast dynamics arise as transition orbits, since the transition holds close to $y=0$ when $H(t_0)<G(t_0)$, and close to $x=0$ when $H(t_0)<G(t_0)$. GSPT [13, 15] could provide tools to address this question. Finally, we give an example to illustrate Theorem 3.7. Consider the system

$$x' = x\mu(t), \qquad y' = y\nu(t), \qquad t' = \varepsilon, \tag{3.24}$$

where μ and ν are given by

$$\mu(t) = \begin{cases} 2(t+1) & \text{if } t \le -1 \\ (t+1)/2 & \text{if } t \ge -1 \end{cases}, \qquad \nu(t) = \begin{cases} (t-1)/2 & \text{if } t \le 1 \\ 2(t-1) & \text{if } t \ge 1 \end{cases}$$

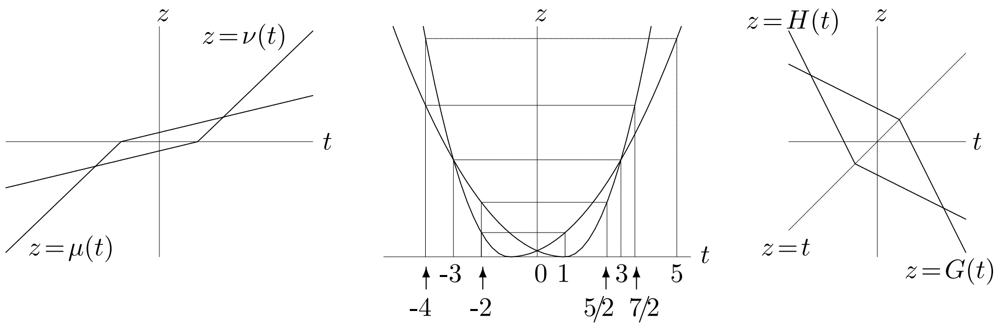

FIGURE 8. The functions μ , ν of (3.24) and the related functions H and G showing that $G(-4)=7/2<5=H(-4)$ and $G(-2)=5/2>1=H(-2)$.

The mapping H and G are given by (see Fig. 8)

$$H(t) = \begin{cases} -2t-3 & \text{if } t \le -1 \\ -(t+3)/2 & \text{if } t \ge -1 \end{cases}, \qquad G(t) = \begin{cases} (3-t)/2 & \text{if } t \le 1 \\ 3-2t & \text{if } t \ge 1 \end{cases}$$

Hence we have $H(t) > G(t)$ if and only if $t<-3$ or $t>3$. According to Theorem 3.7, for all $t_0<-3$ the solution of (3.24) with initial condition $x(t_0)=x_0$ and $y(t_0)=y_0$ remains near the canard solution $x=y=0$ as long as $t_0<t<G(t_0)$ and jumps far from $(0,0)$ along the orbit $x=0$. On the other hand for all $t_0 \in]-3,-1[$ the solution of (3.24) with initial condition $x(t_0)=x_0$ and $y(t_0)=y_0$ remains near the canard solution $x=y=0$ as long as $t_0<t<H(t_0)$ and jumps far from $(0,0)$ along the orbit $y=0$. These results are confirmed by the explicite solutions of (3.24), see Fig. 9:

$$x(t,\varepsilon) = x_0 e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\mu(s)ds}, \qquad y(t,\varepsilon) = y_0 e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\nu(s)ds}.$$

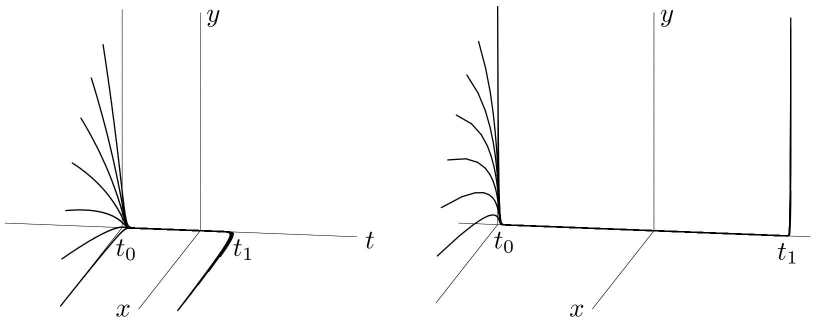

FIGURE 9. Explicit solutions of (3.24) with $\varepsilon = 0.05$. On the left is the case $t_0 = -2$, $t_1 = H(t_0) = 1$ showing that the solution jumps far from the canard solution near the plane $y = 0$. On the right is the case $t_0 = -4$, $t_1 = G(t_0) = 7/2$ showing that the solution jumps far from the canard solution near the plane $x = 0$.

4. PERSISTENCE

Let $\Gamma_\varepsilon = \{\gamma(t, \epsilon) : t \geq 0\}$ the positive semi orbit corresponding to the solution $\gamma(t, \epsilon)$ of (2.1) with initial condition $x(0, \varepsilon) = x_0 > 0$, $y(0, \varepsilon) = y_0 > 0$ and $E(0, \varepsilon) = E_0 > 0$. Our aim is to determine the limit of Γ_ε as $\varepsilon \to 0$. In what follows, we denote by $[\alpha, \beta]$ both interval $[\alpha, \beta]$ in the case where $\alpha \leq \beta$ or interval $[\beta, \alpha]$ in the case where $\alpha \geq \beta$. Let γ_0 is the positive semi orbit corresponding to the solution of (2.2) starting at (x_0, y_0).

4.1. Asymptotic behavior for all $t \geq 0$.

We begin with the asymptotic behavior of the solutions when $E_0 < a$.

Proposition 4.1. *Let* $E_0 \in]0, a[$. *Then* $\lim_{\varepsilon \to 0} \Gamma_\varepsilon = \gamma_0 \cup \gamma_1$, *where*

$$\gamma_1 = \{(\xi_1(E), \eta_1(E), E) : E \in [E_0, E_\infty]\}.$$

Proof. For all $E \in [0, a]$, the equilibrium $(\xi_1(E), \eta_1(E))$ of (2.2) is GAS. Moreover, by assumption **(A3)**, the solutions of (2.5) are converging towards $E = E_\infty$, which is a GAS equilibrium. The result follows from Tikhonov's theory. $\square$

Assume now that $E_0 \in [a, b]$. From Theorem 3.4, we see that the solution $\gamma(t, \varepsilon)$ reaches the neighborhood of point $(\xi_1(E_1), \eta_1(E_1), E_1)$. Then, as shown in Proposition 4.1, it is approximated by a solution of (2.5). More precisely, we have the following result:

Proposition 4.2. *let* $E_0 \in [a, b]$ *and* $E_1 = F(E_0)$. *Then* $\lim_{\varepsilon \to 0} \Gamma_\varepsilon = \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \gamma_3$, *where*

$$\gamma_1 = \{(0, \eta(E), E) : E \in [E_1, E_0]\}, \quad \gamma_3 = \{(\xi_1(E), \eta_1(E), E) : E \in [E_1, E_\infty]\},$$

and γ_2 *is the orbit of (2.2) connecting* $(0,\eta(E_1),E_1)$ *to* $(\xi_1(E_1),\eta_1(E_1),E_1)$.

Proof. Apply Theorem 3.4. and Proposition 4.1 (see Fig 5). □

Assume that $E_0 > b$. From Theorems 3.5 and 3.6 we see that the solution $\gamma(t,\varepsilon)$ reaches the neighborhood of point $(\xi(E_2),\eta(E_2),E_2)$. Then , as shown in Proposition 4.1, it is approximated by a solution of (2.5). In the next propositions, we set two results, in the case where $H(E_0) < G(E_0)$ and in the case where $H(E_0) > G(E_0)$.

Proposition 4.3. *Let* $E_0 > b$. *We assume that* $H(E_0) < G(E_0)$. *Let* $E_1 = G(E_0)$ *and* E_2 *defined by (3.16) in the case where* $E_0 \neq b_0$ *and* $E_2 = E_1 = b_1$ *in the case where* $E_0 = b_0$. *Then* $\lim_{\varepsilon\to 0}\Gamma_\varepsilon = \gamma_0\cup\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_5$, *where*

$$\gamma_1 = \{(0,0,E) : E \in [E_1,E_0]\}, \quad \gamma_2 = \{(0,y,E_1) : y \in [0,\eta(E_1)]\},$$

$$\gamma_3 = \{(0,\eta(E),E) : E \in [E_2,E_1]\} \quad \gamma_5 = \{(\xi_1(E),\eta_1(E),E) : E \in [E_2,E_\infty]\},$$

and γ_4 *is the orbit of (2.2) connecting* $(0,\eta(E_2),E_2)$ *to* $(\xi_1(E_2),\eta_1(E_2),E_2)$.

Proof. Apply Theorem 3.5. and Proposition 4.1 (see Fig. 4 or 6). □

Proposition 4.4. *Let* $E_0 > a_*$. *We assume that* $H(E_0) > G(E_0)$. *Let* $E_1 = G(E_0)$ *and* E_2 *defined by (3.18) in the case where* $E_0 \neq a_0$ *and* $E_2 = E_1 = a_1$ *in the case where* $E_0 = a_0$. *Then* $\lim_{\varepsilon\to 0}\Gamma_\varepsilon = \gamma_0\cup\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_5$, *where*

$$\gamma_1 = \{(0,0,E) : E \in [E_1,E_0]\}, \quad \gamma_2 = \{(x,0,E_1) : x \in [0,\xi(E_1)]\},$$

$$\gamma_3 = \{(\xi(E),0,E) : E \in [E_2,E_1]\}, \quad \gamma_5 = \{(\xi_1(E),\eta_1(E),E) : E \in [E_2,E_\infty]\},$$

and γ_4 *is the orbit of (2.2) connecting* $(\xi(E_2),0,E_2)$ *to* $(\xi_1(E_2),\eta_1(E_2),E_2)$

Proof. Apply Theorem 3.6. and Proposition 4.1 (see Fig. 7). □

4.2. **Practical semi global asymptotic stability.** When $b_1 = 0$ and $a_1 = 0$, the limit

$$\lim_{t\to+\infty,\varepsilon\to 0}(x(t,\varepsilon),y(t,\varepsilon),E(t,\varepsilon)) = (x_\infty,y_\infty,E_\infty), \tag{4.1}$$

is uniform with respect to the initial condition in any compact subset of the positive cone C^3 [7]. This means that equilibrium S is practically semi-globally asymptotically stable. The notion of practical semi-global asymptotic stability in systems depending on parameters, which is very important for applications, appeared first in control theory and is related to the well known problem of stabilization [8].

In the case where $b_1 > 0$, (2.8) is an equilibrium of (2.1) and some solutions could stay near this equilibrium for a very long time so that the limit (4.1) would not be uniform. Indeed, assume that $H(b_0) < G(b_0)$. Then, according to Theorem 3.3, the solution of (2.1) with initial condition $E(0,\varepsilon) = b_0$ will jump quickly near the slow curve $\mathcal{S}_1$ and remains close to this curve, as long as $b_1 < E < b_0$. Then, the solution will jump quickly from the neighborhood of $(0, 0, b_1)$ to the neighborhood of $(0, \eta(b_1), b_1)$ and stay for a long time near this equilibrium before jumping towards $(\xi_1(E_1), \eta_1(b_1), b_1)$.

Similarly, in the case where $a_1 > 0$, (2.9) is an equilibrium of (2.1) and some solutions could stay near this equilibrium for a very long time so that the limit (4.1) would not be uniform. Indeed, assume that $H(a_0) > G(a_0)$. Then, according to Proposition 3.3, the solution of (2.1) with initial condition $E(0, \varepsilon) = a_0$ will jump quickly near the slow curve $\mathcal{S}_1$ and remains close to this curve, as long as $a_1 < E < a_0$. Then, the solution will jump quickly from the neighborhood of $(0, 0, a_1)$ to the neighborhood of $(\xi(a_1), 0, a_1)$.

5. Applications to Clarks's model

In this section we consider system (1.2). Assume that all parameters of (1.2) are positive.

The fast dynamics are illustrated in Fig. 10. We have

$$a = c = \frac{r}{q_1}, \qquad b = \frac{s}{q_2},$$

$$\xi_1(E) = \xi(E) = K\left(1 - \frac{q_1}{r}E\right), \quad \eta_1(E) = \eta(E) = L\left(1 - \frac{q_2}{s}E\right).$$

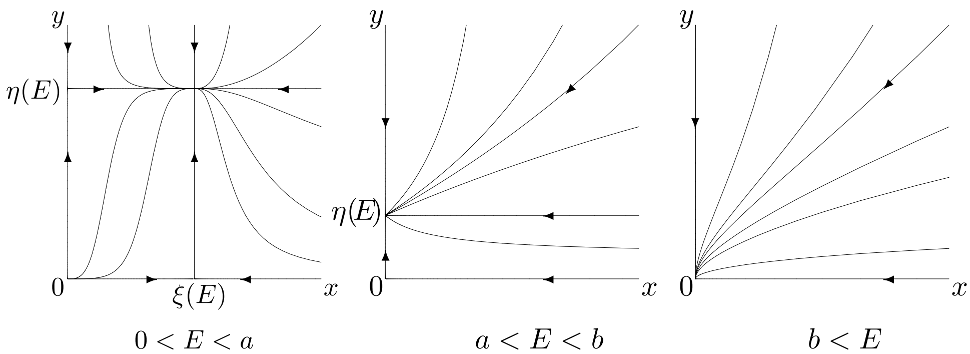

FIGURE 10. The fast dynamics of system (2.2). The equilibrium $(0, 0)$ is attracting for $E > b$. The equilibrium $(0, \eta(E))$ is attracting for $a < E < b$. The equilibrium $(\xi(E), \eta(E))$ is attracting for $0 < E < a$.

The slow manifold is represented in Fig. 11. Assumptions **(A1)** and **(A2)** hold if and only if

$$sq_1 > rq_2. \tag{5.1}$$

On the other hand, the subset $\Pi = \{(x, y) : p_1 q_1 x + p_2 q_2 y = c\}$ intersects $\Gamma = \{(\xi(E), \eta(E)) : 0 < E < a\}$ at a unique point, so assumption **(A3)**

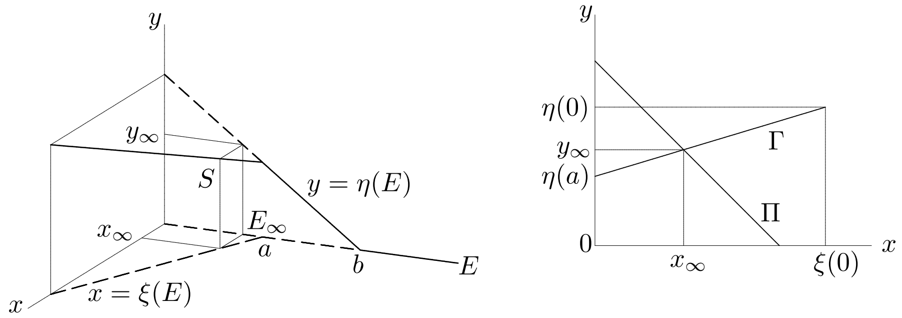

Figure 11. On the left: the slow manifold of system (1.2). Attracting parts of the slow manifold are indicated by a bold line, non attracting parts of the slow curve are indicated by a dashed line. On the right: the relative positions of Γ and Π. For the set of parameters (5.3) we have $P(\xi(0),0)>0$.

holds, if and only if (see Fig. 11, right)

$$\frac{sq_1 - rq_2}{sq_1} p_2q_2L < c < p_1q_1K + p_2q_2L. \tag{5.2}$$

The equilibrium $S = (x_\infty, y_\infty, E_\infty)$ is given by

$$x_\infty = (sq_1c - (sq_1 - rq_2)p_2q_2L)\frac{K}{\Delta}, \quad y_\infty = (rq_2c + (sq_1 - rq_2)p_1q_1K)\frac{L}{\Delta},$$

$$E_\infty = (p_1q_1K + p_2q_2L - c)\frac{rs}{\Delta}, \quad \text{where } \Delta = sp_1q_1^2K + rp_2q_2^2L.$$

We have the following result.

Lemma 5.1. *Under condition (5.1) we have $H(E) < G(E)$, for all $E > b$.*

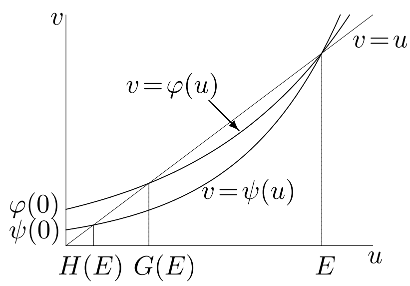

Figure 12. For (1.2), the functions H and G satisfy $H(E) < G(E)$ for all $E > b$.

Proof. We have

$$h(E) = \int_a^E \frac{r - q_1u}{-cu} du = \frac{q_1E - r\ln E - q_1a - r\ln a}{c}.$$

Let $E > b$ and $u = H(E)$. We have $q_1 u - r\ln u = q_1 E - r\ln E$. Thus,

$$u = \varphi(u), \text{ where } \varphi(u) := Ee^{\frac{u-E}{a}}.$$

Hence $H(E)$ is a fixed point of the function $v = \varphi(u)$. We have

$$\varphi(0) > 0, \quad \varphi(E) = E, \quad \varphi'(E) = \frac{E}{a} > 1 \text{ and } \varphi''(u) > 0.$$

Hence the function $v = \varphi(u)$ has a unique fixed point $u = H(E)$ in $]0, E[$. Similarly we have

$$g(E) = \int_b^E \frac{s - q_2 u}{-cu} du = \frac{q_2 E - s\ln E - q_2 b - s\ln b}{c}.$$

Thus $u = G(E)$ is a fixed point of the function $v = \psi(u) := Ee^{\frac{u-E}{b}}$. We have

$$\psi(0) < \varphi(0), \quad \psi(E) = \varphi(E) = E \text{ and } \frac{1}{a} > \frac{1}{b}.$$

Hence the functions φ and ψ satisfy $\psi(u) < \varphi(u)$ for all $u \in]0, E[$, so that $H(E) < G(E)$, see Fig. 12 □

Thus, under conditions (5.1) and (5.2), the asymptotic behavior of the solutions is given by Propositions 4.1, 4.2 and 4.3. The case depicted in Proposition 4.4 does not occur since we have $H(E) < G(E)$ for all $E > b$. To illustrate our results, we carried some numerical experiments with $\varepsilon = 0.01$ and the following set of parameters:

$$K = 5, \quad L = 3, \quad r = 1, \quad s = 2, \quad p_1 = p_2 = q_1 = q_2 = 1, \quad c = 4. \quad (5.3)$$

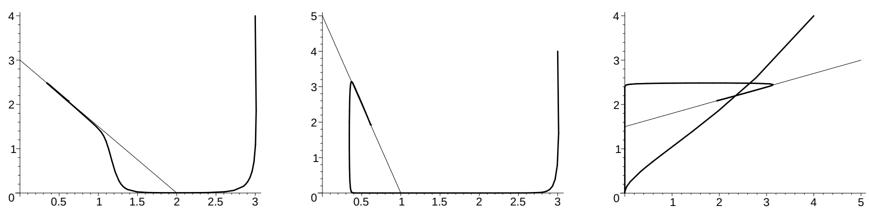

FIGURE 13. Numerical solutions of (1.2) with $x_0 = 4$, $y_0 = 4$ and $E_0 = 3 < a_*$.

For clarity, we draw the projections of the trajectories, related to two sets of initial values (see Fig. 13 and Fig. 14), on the planes (E, y), on the left, (E, x), in the center and (x, y), on the right. The behavior of the first trajectory with $x_0 = 4$, $y_0 = 4$ and $E_0 = 3 < a_*$ is in accordance with the results of Theorem 3.5 and Proposition 4.3. The behavior of the trajectory with $x_0 = 0$, $y_0 = 0$ and $E_0 = 6 > a_*$ supports the results of Theorem 3.5 and Proposition 4.3.

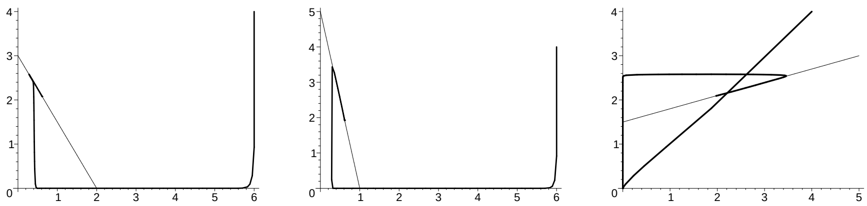

FIGURE 14. Numerical solutions of (1.2) with $x_0 = 4$, $y_0 = 4$ and $E_0 = 6 > a_*$.

Acknowledgement. The authors thank the anonymous referee for fruitful suggestions and, in particular, for his indication of how to solve the problem locally using the center manifold reduction. HB thanks Mulhouse University for its hospitality during the completion of this work. TS thanks D. Panazzolo for helpful explanations on geometric singular perturbation theory and center manifold reduction.

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